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SOLVING STRUCTURAL MECHANICS PROBLEMS ON DIGITAL COMPUTERS

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H. J. Greenberg

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Institute of Mathematical Sciences

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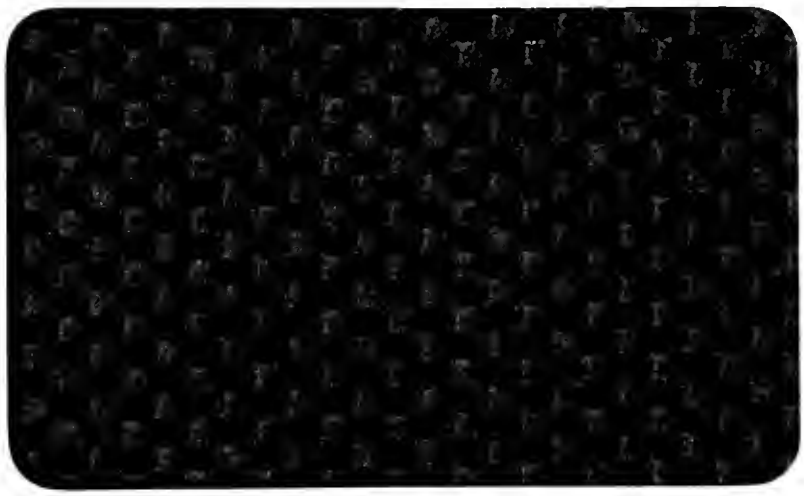

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ABSTRACT

This paper constitutes a short, critical survey of the current applications of digital computers to solving problems in structural mechanics. It includes a brief description of typical medium and large size computing systems (IBM-650 and IBM-704), a list of standard mathematics problems to which structural mechanics problems commonly reduce for which adequate codes exist together with a list of these codes for both machines, an informal survey of mechanics codes completed and under development at representative centers, a detailed discussion of three advanced computer applications and concludes with recommendations for future work in this field.

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SOLVING STRUCTURAL MECHANICS PROBLEMS ON DIGITAL COMPUTERS

Introduction.

The purpose of this paper is to give a critical but abridged survey of the current use of digital computers for the solution of problems in structural mechanics. It was not possible for the present conference to produce a complete bibliographic review of all the current work being done in this area, the sheer volume is too great despite the fact that the first electronic automatic computer, the ENIAC only dates back to 1946. At the same time, much of this work, although useful, might be classified as routine, in which existing mathematical subroutines are strung together to effect a numerical solution of a problem by a method which is a more or less direct transcription of standard procedures of structural mechanics which have been used in the past as the basis for slide rule or hand computer calculations. These applications will merely be mentioned briefly or omitted altogether. The only applications considered here in detail are of a more advanced character both in the mechanics and in the mathematics.

In Section A there is a brief review of the IBM-650 and 704 systems as typical examples of intermediate and large size

computers. There follows a list of mathematics problems commonly encountered in structural mechanics and a specific code is cited for each of these problems for each machine. The purpose of this is to furnish a comparison of capacities and speeds, to indicate the present state of the art in mathematical subroutines and to possibly aid the reader in selecting a code for a problem.

The remainder of Section A is devoted to the results of an informal survey of the computer applications in structural mechanics being carried out at a number of universities, government laboratories, industrial laboratories and engineering consultant firms, with emphasis on the development of general codes.

In Section B three particular computer applications are studied in some detail. These are

(1) Non-linear deflections of shallow spherical shells.

(2) Non-linear bending and buckling of circular plates.

(3) Plastic torsion of cylinders.

The final section, contains some conclusions and suggestions for future work.

A. Machines. Problems, and Programs.

1. Digital computers. It is appropriate perhaps to begin with a brief discussion of machines. A number of makes and models are in current use[1]; however, for the purposes of this paper attention will be focused on systems of two sizes: the large size as typified by the IBM-704 and the intermediate size as typified by the IBM-650. The nature and capabilities of these machines is assumed to be sufficiently well-known so that only a bare recital of statistics is needed. Comparison of these computers on specific problems will follow later and will sharpen an appreciation of their capacities and limitations.

The basic IBM-650 possesses a magnetic drum memory of 2000 words. Computations are performed in the decimal system and there are 10 digits to the word. Average access time to a memory location is 2000 microseconds.

The IBM-704 has a magnetic core type memory usually of 8,192 words but also is available with 16,384 or 32,768 words, each 36 binary (approximately 10 decimal digits) bits in length. Addition or subtraction requires 24 microseconds and multiplication or division is accomplished in 240 microseconds.

Programming for these machines can be carried out at either of two levels. At the level of the more expert coder,

[1] For a complete listing, see [1]. For a short history of digital computers see [2].

symbolic code language is used, which itself assembles a machine language program on the machine. This requires substantial training and knowledge of the machine. Such codes are, for example, SAP (SHARE ASSEMBLY PROGRAM) for the 704 and SOAP (SYMBOLIC OPTIMAL ASSEMBLY PROGRAM) for the 650. At the level of the inexperienced machine user who wishes to devote a minimum amount of time in learning to code, there are automatic coding systems which more closely resemble the language of mathematics and are designed primarily for scientific and engineering computation. Among these are IT (INTERNAL TRANSLATOR) for the 650 and FORTRAN (FORMULA TRANSLATOR) for the 704. These codes are concise to write and themselves generate an efficient symbolic code which in turn generates the machine language code. There is a so-called FOR TRANSIT system which makes FORTRAN available to 650 users by translating FORTRAN statements into IT.

2. Mathematics problems and programs. Problems in structural mechanics frequently reduce to one or another standard mathematics problem of a type for which adequate programs already exist in the library of subroutines for the machine in use. Although the author wishes in large measure to discourage the "as is" use of existing codes, which in many cases would be inefficient or inadequate,

particularly for larger problems, still there are numerous problems where it is most convenient to use available programs.

Through an organization of users such as SHARE for the 704 and the IBM-650 Program Library codes are exchanged and distributed. The libraries of codes available have become so voluminous that the person interested in selecting a program for solving a given problem needs the guidance furnished by the experience of the "old-hands" at his particular computing center.

Typical and perhaps most common among the standard mathematics problems to be solved in structural mechanics for which good programs are available are those of

(i) Matrix inversion and solution of a system of linear algebraic equations

(ii) Finding the eigenvalues and eigenvectors of a matrix

(iii) Integrating a system of first order ordinary differential equations

(iv) Linear programming.

Problems involving the solution of partial differential equations leading through difference approximations to large systems of linear (or non-linear) algebraic equations are common today but in general one must select a method and write a code tailored to the individual problem. Examples

of these will be given in Section B.

In the following, specific codes are cited for the 650 and 704 for the problems listed above. This is done to furnish a comparison of the capacities and speeds of the machines, to indicate something of the present state of the art, and to possibly assist the reader in the selection of a program for his problem. Although these codes are considered among the best, there are others equally good both for these and other machines.

(1) Matrix Inversion and simultaneous linear equations[2]. A code by D. W. Sweeney [4] for the IBM-650 inverts matrices of order ≤ 42 or solves b sets of simultaneous equations with $n^2 + nb \leq 1764$. The matrix elements are given in floating-point form. Gaussian elimination is followed and coefficients from the diagonal are used regardless of size, (which can cause trouble). The matrix inversion is carried out in approximately $.072\ n^3$ seconds plus input-output time. Thus, for $n = 40$, the time required for the calculations is over $1\ 1/4$ hours.

For the IBM-704, an NYU program [5] based on the Los Alamos LAS 665 code available through SHARE and using Gaussian elimination will calculate the inverse of a 40 x 40 matrix in 36 seconds and of a 60 x 60 matrix in

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[2] For a general survey of methods and bibliography, see [3].

2 minutes. The instructions for this code occupy some 225 words of memory so that in principle the order of the system which can be solved is limited only by the need to store coefficients in the remainder of the memory. As far as storage is concerned, therefore, in an 8,192 word 704, a matrix problem of order 89 could be attempted. However, success in terms of accuracy in the answer is dependent upon the condition of the matrix. Thus, for the ill-conditioned Hilbert matrix with elements $h_{ij} = (i+j-1)^{-1}$, $i,j = 1, \ldots, n$, for $n = 4$ with a determinant whose value is 1.65×10^{-7} about 5 significant digits can be obtained accurately in the inverse matrix. At $n = 9$ with a determinant 9.72×10^{-43} in value, the code fails to give any digits accurately. A comparatively well conditioned matrix such as Γ_n with elements $\gamma_{ij} = i(n+1-j)/(n+1)$, $i \leq j$, $\gamma_{ji} = \gamma_{ij}$, $i,j = 1, \ldots, n$ gives for $n = 60$, with a determinant of 1.64×10^{-2} about 5 significant figures accurately in the time of two minutes previously quoted.

(ii) Eigenvalues and eigenvectors of a real symmetric matrix.[3] A code developed at Indiana University for the 650 (IBM-650 Library Program Abstract not yet distributed) based on the Jacobi matrix diagonalization procedure will give all eigenvalues and vectors for $n \leq 32$ in a time of approximately $(2.5 \times 10^{-4})n^4 + 4 \times 10^{-3}\, n^3$ minutes and eigenvalues

[3] For a survey of methods and bibliography, see [6] and [3].

only for $32 < n \leq 56$ in a time of approximately $0.006n^3$ minutes plus punch out time. Thus, to calculate all eigenvalues and vectors for $n = 30$ would take about 3 hours and 20 minutes.

For the 704, an NYU code (NU MXEV - available through SHARE) based on Given's method [7] will handle an 80 x 80 matrix, given 8192 words of memory, and a 120 x 120 matrix, given 16,384 words of memory. In fixed point arithmetic (optional) the time in seconds in the range $20 \leq n \leq 50$ is given as $.3n + .075\, n^2$ to give both eigenvalues and vectors. For $n = 30$ the time would be approximately 1 1/4 minutes. Loss of accuracy with increasing n is felt first in the vectors then in the eigenvalues. For $n = 120$, in floating point, double precision arithmetic would probably be required just to get the largest eigenvalue.

(iii) Ordinary differential equations. For the IBM-650, R. W. De Sio has prepared a code [8] for the solution of a system of first order equations $y_i' = f_i(x, y_1, \ldots, y_n)$, $i = 1, \ldots, n$, subject to given initial values at $x = 0$. This is a fixed-point routine based on a predictor-corrector method that is repeated until the desired convergence is obtained at each step. The storage required is 380 locations plus what is required for the calculation of the function f_i, which in turn would determine the running time per x increment.

For the IBM-704, the NYU code NU DEQ [9] can be used to integrate the above system. The method is based on a variation by S. Gill [10] of the Runge-Kutta procedure. This is a floating-point code using 110 storage locations for the instructions exclusive of the calculation of the f_i.

(iv) Linear programming. This is a newer and less standard type of problem than the preceding, but one which is rapidly growing in importance. We shall refer to a problem of this type later as well as a problem in quadratic programming occurring in structural mechanics. The linear programming problem in general is that of minimizing or maximizing a linear expression in a set of unknowns $x_1,\ldots,x_n$, subject to a number m of side conditions which consist of either linear algebraic equations or inequalities involving the unknowns. Unrestricted variables (as to sign) can always be replaced by non-negative variables by introducing new unknowns. Inequalities similarly can always be replaced by equations by increasing the number of unknowns. The codes cited next assume these changes to be made.

For the IBM-650, there is a code by L. S. Woo [11] based on the Modified Simplex Method [12] which handles a maximum of 97 equations in an unlimited number of variables. The time required is stated to vary from 4 to 13 minutes per iteration during the first 40 iterations.

For the IBM-704 a code by Orchard-Hayes, Judd and

Cutler, available through SHARE, also based on the Modified Simplex Method, will handle up to 255 equations in any number of variables. A general estimate of time required is not available; however, it is stated that an example involving 50 equations and 100 variables required 20 minutes for solution.

3. Structural mechanics problems. In an effort to ascertain what use is currently being made of digital computers for the solution of structural mechanics problems, the author conducted an informal survey of a number of university, government and industrial sites where computing is done on these problems. The list included Brown University, M.I.T., University of Illinois, University of Houston, New York University, National Bureau of Standards, David Taylor Model Basin, Aberdeen Proving Grounds, Boeing Airplane Company, Gulf Oil Corporation, IBM Corporation, General Electric Research Laboratories, General Electric Flight Propulsion Laboratory Department, Westinghouse Research Laboratories, M. W. Kellogg Company, American Bridge Division of U. S. Steel, Gibbs and Cox, and the firm of Howard, Needles, Tammen and Bergendoff. This list could have been extended indefinitely but is considered to be a fair sample.

Before reviewing the technical information obtained

in the course of this survey, it may be appropriate to consider the need for exchange of information of this kind. The number of centers and the number of individuals involved in writing programs for the express purpose of solving problems in structural mechanics is multiplying so rapidly that any static attempt to compile a bibliography or listing of codes would be of value for only a short time and would likely be outdated before it appeared in print. Accordingly it seems clear that what is needed is an organization of interested groups to provide a mechanism for the continuing interchange of codes and other information pertaining to the solution of structural mechanics problems on digital computers. This could be accomplished through the publication by such an organization of a periodic newsletter, for example, containing reports sent in on a voluntary basis by members on their codes and activities. Representatives of the cooperating groups could meet periodically in conjunction with the regular meetings of an appropriate technical society such as the Applied Mechanics Division of the A.S.M.E. Precisely this sort of organization has been set up, for example, for the exchange of nuclear reactor codes and is known as the Nuclear Codes Group (NCG). Its newsletter is distributed through one of the member groups, the AEC Computing and Applied Mathematics Center at New York University. This organization of users

has been assisted by IBM which acts to distribute decks of cards as requested by members for actual codes for IBM machines which have been submitted to the NCG.

Returning now to the survey of structural mechanics codes, it is perhaps most instructive for the purposes of the present discussion to describe the situation in general terms, as it appeared, rather than to attempt to list specific codes.

Obviously every paper that appears in a journal giving numerical results for a specific problem and acknowledging the use of a computer in itself constitutes a computer application. However, one may make a general distinction between one-shot applications, or codes that serve only once, and general codes that are designed to cover a whole class of problems. The first type is noteworthy when a new method is developed or the application is unusual. It is the second type of code, however, that is of the most interest to us here. It appears that people are now in the first phases of writing such general purpose codes for structural mechanics.

Starting with the mathematically simplest problems, we find that there are codes for the elastic stress analysis of two and three dimensional trusses and rigid frames[4] under

[4] Extensive work on computer analysis of such structures has been carried out by Livesley in England. See, for example, [13] and also bibliography in [17].

static loads (University of Houston, Gulf Oil Corporation, American Bridge Division of U. S. Steel among others). Related to these are codes for the analysis of thermal expansion stresses in arbitrary three dimensional piping systems (M. W. Kellogg Company among others). There is a code for the elastic-plastic analysis of trusses calculating both stresses and displacements up to the plastic limit load (New York University). Ordinary plastic limit analysis of trusses and framed structures can be carried through using standard linear programming codes [14]. A code has been developed for the minimum weight design of frames (Brown University [15]). Codes for the elastic-plastic response to air blast loads of such structures as beams, circular plates, rectangular membranes have been prepared at the Ballistic Research Laboratory at Aberdeen Proving Grounds.

A number of codes for the solution of structural problems of especial concern to the civil engineer have been developed at the University of Illinois for the ILLIAC under the general direction of N. M. Newmark. As reported on by Veletsos [16], these include the stress analysis of simply supported skew I-beam bridges, determination of natural frequencies and modes for the vibration of multi-story building frames with flexible girders, dynamic response of building frames under wind pressures, blast loads and ground disturbances, and the dynamic response of simple span highway bridges under the

passage of heavy vehicles, among others. Bridge analysis programs have also been developed at the University of Houston. In general the civil engineering field has been active in the development and use of computer codes for analysis and design for several years[5]. As an indication of this activity we cite the May 1958 issue of Civil Engineering which is entirely devoted to digital computers and their applications. According to this source, over 100 civil engineering firms and 40 state highway departments have their own computers. A Committee on Electronic Computation of the Structural Division of the A.S.C.E. is currently conducting a survey of computer codes.

In the aircraft industry there are general codes for flutter analysis and the determination of the natural frequencies and modes of delta wings and "built-up" structures formed from plates, bars and stringers (Boeing Aircraft among others).

The analysis of cylindrical shells and shells of revolution is of widespread interest in such connections as the design of submarines, missiles, turbines, reactors. A code for the calculation of the strength of stiffened cylindrical shells was developed at the David Taylor Model Basin. At the General Electric Gas Turbine Division several

[5] R. W. Clough [17] has made a survey of structural analysis applications of computers with a good bibliography.

codes have been developed for the small deflection theory of shells under axially symmetric and also unsymmetric loadings. Of considerable interest is a G.E. code which will reportedly analyze a built up structure of three shells of revolution joined in series. The heart of the G.E. shell programs is a routine to solve systems of 8^{th} order linear ordinary differential equations.

Methods and codes for the solution of non-linear bending and buckling problems in elastic plates and shells have been developed at New York University and M.I.T. Some of these developments will be reviewed later in this paper as special examples.

Codes have been and are being developed for the solution of 4^{th} order biharmonic type boundary value problems (NYU and M.I.T.) and problems associated with 2^{nd} order quasi-linear partial differential equations (NYU). The former have application in two-dimensional elasticity problems including plane strain, plane stress and plate problems. The latter has various applications in mechanics, for example in plastic torsion with strain-hardening, which will be reviewed later as a special example.

B. Some Case Histories.

The birds-eye view of the survey of the last section is of necessity lacking in detail. To compensate for this, the present section is a report on three specific computer applications with which the author is familiar. All are currently being carried out at the A.E.C. Computing and Applied Mathematics Center at New York University. It is believed that these applications are interesting, first, because they are concerned with problems of a difficult and advanced character, both mechanically and mathematically and, second, because of the light they shed on the various roles which the computer can play in research.

1. Non-linear deflections of shallow spherical shells[6]. The mechanical problem is the determination of the deflections of a thin spherical cap under uniform external pressure and clamped around the edge. To account for experimentally observed loads below those predicted by the linear theory of buckling and the snap-through form of buckling mode the problem must be treated as one in the non-linear theory of buckling following von Karman, Tsien and Friedrichs. In [18] the problem is reduced to solving the following pair of non-linear second order ordinary differential equations

[6] For a complete report see [18] including references cited in the present discussion.

$$L\,\alpha = \rho\,(\alpha\gamma + P\theta^2)$$
$$L\,\gamma = \rho\,(\theta^2 - \alpha^2) \qquad (1)$$

where $L \equiv \theta \frac{d}{d\theta} \frac{1}{\theta} \frac{d}{d\theta} \theta$ is the differential operator, θ, the ratio of the polar angle to the total angular opening is the independent variable, P is a loading parameter, ρ is a geometric parameter proportional to the ratio of the shell rise to thickness, $\alpha = \alpha(\theta)$ is proportional to the slope of the shell, $\gamma = \gamma(\theta)$ is a stress function and all quantities are dimensionless. Deflections and stresses are obtained from α and γ. The boundary conditions are $\alpha = \gamma = 0$ at the center, $\theta = 0$, and $\alpha = 1$, $\frac{d\gamma}{d\theta} - \nu\,\gamma = 0$ at the edge, $\theta = 1$, where ν is Poisson's ratio.

Although solutions for this problem had been obtained in the past by perturbation and by power series methods, they unfortunately covered only a small range of the parameters. The problem, therefore, was to devise a method of numerical solution for machine calculation which would cover a wide range of parameters, from very shallow shells (near plates) to shells with a high rise, from low loads to buckling loads (in some way to be determined) to perhaps post-buckling loads and the possible numerical determination of non-adjacent equilibrium configurations. In other words what was desired was a code which would enable one to study the behavior of a whole range of shells under widely varying loads.

A power series method of solution[7] was attempted first and carried through as follows. Inserting expansions of the form

$$\alpha = \sum_{n=1}^{\infty} \alpha_n \theta^{2n-1} , \qquad \gamma = \sum_{n=1}^{\infty} \gamma_n \theta^{2n-1} \tag{2}$$

into (1), recurrence relations were obtained for the α_n and γ_n in terms of the initial coefficients α_1 and γ_1. The boundary conditions at the edge reduced to

$$\sum_{n=1}^{\infty} \alpha_n - 1 = 0 , \qquad \sum_{n=1}^{\infty} [2n - (1+\nu)\gamma_n] = 0 \tag{3}$$

which were in effect then simultaneous equations for α_1 and γ_1 for given values of the parameters ρ and P. For fixed values of ρ and P, the numerical procedure consisted in at first choosing trial values for α_1 and γ_1, calculating successive coefficients α_n and γ_n in the series from the recurrence relations until a convergence criterion was satisfied, and then evaluating the functions in (3) to see whether these equations were satisfied to a required degree of accuracy. Improved values of α_1 and γ_1 were obtained systematically by a Newton-Raphson type of iteration toward the solution of the equations (3). When these equations were satisfied the displacements and stresses were computed from the power series expressions evaluated from the final values of α_n and γ_n obtained.

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[7] Solutions by power series on a computer have also been obtained for this problem at M.I.T. by Weinitschke [19] under Reissner.

For a given shell (ρ value), the computations were begun with a small load (P value) for which the deformed state was nearly spherical so that a good initial estimate of the α_1, γ_1 values could be made. The Newton-Raphson process then rapidly gave the roots α_1 and γ_1 of (3) to the desired accuracy. For successively higher values of P, the initial guess for α_1 and γ_1 was found by extrapolation of roots obtained for preceding P values. This was aided by a plot of points in an α_1, γ_1-plane for increasing P values for each shell.

The calculations were carried out on the AEC UNIVAC at New York University using a code of some 300 words in length. It did not prove practical to have the initial α_1, γ_1 guessing programmed due to extreme sensitivity to correct α_1, γ_1 choices especially for larger values of ρ and P. Human judgement was needed and close cooperation between man and machine was required for good results.

For a given shell, the load was increased in small increments and solutions obtained according to the described procedure. The loading process terminated by a lack of convergence in one of two ways. Either the Newton-Raphson process failed to converge (convergence to an accuracy of 1×10^{-4} in (3) was usually obtained in under 10 iterations if at all) or else the higher coefficients α_n, γ_n calculated from assumed α_1, γ_1 values failed to converge (i.e. to

become less than 1×10^{-4} in 100 terms).

Typically the process ended suddenly. Good convergence was obtained for a succession of increments in P for a given shell. Then for any slight increase in P no solution could be obtained by the process through a selection of α_1, γ_1 values in the neighborhood of one extrapolated from values for slightly lower values of P. It was as evident as possible in a numerical process that no neighboring solution existed to the preceding ones. The buckling load was then assumed to lie between the highest attained value of P for which the process converged and the next larger value attempted. This was completely analogous to the situation in a laboratory test when the buckling load is reached and a new equilibrium configuration, substantially different from the previous ones, is attained. As examples of this numerical "buckling" behavior, for $\rho = 32$ and $P = 1.25$ the procedure converged to a solution in 3 iterations. Increasing P to 1.3, no convergence in the Newton-Raphson process was in evidence after 10 iterations. The buckling load P_{cr} for $\rho = 32$ was therefore assumed to lie between $P = 1.25$ and $P = 1.3$. For $\rho = 26.1$ and $P = 1.55$ convergence occurred in 2 iterations and only 40 terms of the power series were needed, while for $P = 1.6$ neighboring α_1, γ_1 values could not be found to lead to convergent power series even after 99 terms. The buckling load in this case was

assumed to lie between $P = 1.55$ and $P = 1.6$. The results were in good agreement with experiments done by Kaplan and Fung [20]. As a time estimate for solving a given shell under a given load, the time per iteration was roughly one minute in the case where 50 terms in the power series were needed. Thus in the cases cited, 2 or 3 minutes of computation led to a complete listing on tape of the radial deflection and all stress components for the top and bottom surface of the shell as functions of angle for the given load.

Interestingly enough, it was possible for some values of ρ to pick up solutions corresponding to loads in excess of the buckling load. This was not done by continuously following the solutions as the loads are increased from below the buckling load. As explained above this was not possible. However, one could locate the post-buckling branch of the curve in the α_1, γ_1 plane by taking advantage of the two parameter character of the problem. For low enough values of ρ, the shell approximates a flat plate and no buckling occurs so that one may continuously locate solutions up to any load, including those in excess of buckling loads for shells of larger ρ value. Having attained a solution for small ρ and large P, one can then vary ρ keeping P constant and continuously proceed to the solution for values of ρ for which the attained P

is in the post-buckling range. One can then decrease P for fixed ρ and find a sequence of solutions for the loads between the buckling load and attained load. Thus, numerically one could locate different branches of solutions, say in the α_1, γ_1 plane, and from these, different branches of typical load-deformation curves (say for the center of the spherical cap) corresponding to different ranges of loads.

The computations clearly showed certain effects which had not been predicted theoretically before and for which, although experimentally observed, there existed little precise data. These effects were the dependence of buckling mode on ρ, the shape parameter together with an associated non-monotone dependence of buckling load on ρ and the existence of a boundary layer for higher ρ. Thus, it was found, from plots of the radial deflection, that for $\rho < 23$ approximately, the ordinary mode of buckling occurred in which the maximum deflection is at the center and falls to zero at the edge. Increasing ρ still more leads to flattening of this maximum and transition to a second mode where the maximum moves from the center progressively toward the edge, and where rapid changes in deflection and stress define a boundary layer. This mode seems to persist until around $\rho = 52$ where a third mode begins to take shape with two maxima, one at the center. Plots of buckling load against

ρ revealed a peaking behavior with the peaks occurring at the values of ρ where the transitions in mode occurred.

Generally speaking, the power series solution was no longer satisfactory when it came to large values of ρ or large P (post-buckling), due to the necessity for increasingly large numbers of terms in the expansions. It was also dependent upon finding close initial estimates for α_1 and γ_1 and could proceed therefore only by continuous variation of parameters and solutions. The next example discusses a method of "iterative differences" which was applied subsequently to the present shell problem and proved to be much faster and relatively insensitive to initial guesses. Using this technique solutions for a larger range of ρ and P were obtained.

2. Non-linear bending and buckling of circular plates [21,22]. A thin circular elastic plate is considered, subjected to either a uniform lateral pressure or a uniform radial edge thrust with clamped or simply supported edges. The von Karman plate equations in polar coordinates for rotationally symmetric deformations, reduce to

$$\begin{cases} L\,\alpha(x) = -\,\alpha(x)\,\gamma(x) - P\,x^2 \\ L\,\gamma(x) = \frac{1}{2}\,\alpha^2(x) \end{cases} \tag{4}$$

where $L \equiv x\,\frac{d}{dx}\,\frac{1}{x}\,\frac{d}{dx}\,x$ is the second order differential

operator as previously, x is the dimensionless radius, $\alpha(x)$ is a dimensionless slope, $\gamma(x)$ is the dimensionless radial derivative of the Airy Stress Function, and P is the dimensionless lateral load. Various boundary conditions are considered, in particular we may here cite the bending problem (A) with clamped edge, for which

$$\alpha(1) = 0 \ , \quad \left[\frac{d\gamma}{dx} - \nu\gamma\right]_{x=1} = 0 \ ,$$

and the buckling problem (E) with clamped edge, for which

$$\alpha(1) = 0 \ , \qquad \gamma(1) = F \geq 0 \ .$$

In both cases by symmetry and regularity, $\alpha = \gamma = 0$ at the center, $x = 0$.

The numerical method of solution consisted in approximating the equations (4) by finite differences. The resulting system of non-linear algebraic equations for the unknown function at the radial mesh points $x_i = ih$, $i = 1, \ldots, m-1$, were then solved by the following "interpolated" iteration process. Boundary conditions are imposed at the end points $x_o = 0$, $x_m = 1$.

Letting $\bar{L}$ stand for the difference operator corresponding to L a simple iteration process is defined by

$$\left.\begin{aligned} \bar{L}\,\alpha_n(x_i) &= -\,\alpha_{n-1}(x_i)\,\gamma_{n-1}(x_i) - P\,x_i^2 \\ \bar{L}\,\gamma_n(x_i) &= \tfrac{1}{2}\,\alpha_n^2(x_i) \end{aligned}\right\} 0 < i < m \qquad (5)$$

Starting from an initial guess for $a_o(x_i)$, $\gamma_o(x_i)$ the first of equations (5) defines a system of $m-2$ linear equations for $m-2$ unknowns $a_1(x_i)$. From the $a_1(x_i)$, the $\gamma_1(x_i)$ can be found from the second of equations (5) as the solution of a linear system. However, analysis and calculations show that this process will converge only for a small range of the parameters F or P and will definitely diverge for values of F and P above certain predicted values.

An improved procedure (which can be motivated on mechanical grounds) is obtained if one interpolates at the n^{th} step between the previous iterate $a_{n-1}(x_i)$ and the one found as the solution of the first set of equations in (5), and then uses the interpolated function as the new iterate. Thus, one sets

$$a_n(x_i) = \omega\, a_n^*(x_i) + (1 - \omega)\, a_{n-1}(x_i) \qquad (6)$$

where a_n^* denotes the solution of the equations

$$\bar{L}\, a_n^*(x_i) = - a_{n-1}(x_i)\, \gamma_{n-1}(x_i) - P\, x_i^2 .$$

Here ω is the interpolation parameter, which lies between 0 and 1.

The best value to use for ω is determined empirically by setting up a convergence criterion such as

$$\max_{0 \le i \le m} \left\{ |a_n(x_i) - a_{n-1}(x_i)| \,,\ |\gamma_n(x_i) - \gamma_{n-1}(x_i)| \right\} < 10^{-5}$$

and finding the number of iterations $N(\omega)$ required as a function of ω. Typically one finds a graph with a sharp and sensitive minimum which is a function of the parameter value of P or F. This defines the optimum ω value. Fortunately the optimum ω was found to be relatively insensitive to the mesh size so that using a coarse mesh and little computing time one could obtain by machine experiments the proper ω values to be used.

In problem (A), the initial iterate was taken to be the converged solution for the next smaller load value. In problem (E) there is the possibility of more than one solution and the method should be applied with this in mind. For $F < F_o$ (where F_o is the lowest eigenvalue of the linear buckling problem) the only solution is that of direct compression, $\alpha(x_i) = 0$, $\gamma(x_i) = Fx_i$. The value of F_o, however, is not needed à priori. One can start with a non-zero estimate for $\alpha(x_i)$; for $F < F_o$ the iteration will converge to the trivial solution $\alpha(x_i) = 0$. By increasing F in small increments from zero, at some $\bar{F}_o$ the iteration will converge to a non-trivial solution and $\bar{F}_o$ may be taken as an estimate of F_o. From here on as F is increased above $\bar{F}_o$, the previous non-trivial solution is used as the first estimate. When the iterations converge for a given value of the load, the stresses and deflections are computed.

The calculations were carried out on the AEC UNIVAC at New York University, using 10, 20 and 50 mesh points. For 50 points some 500 words of memory were required for the complete code. The method was found to give excellent convergence for a practically unlimited range of loads. In case (A), values of the parameter P as high as $P = 7000$ were easily attained.

For comparison the method of power series previously discussed was also used for the present problem in case (A). The solutions obtained agreed closely with those of the present method. It was however found that the power series method failed for much lower loads, around $P = 2000$. Moreover, the difference-iteration method was faster in general by a factor of ten (thus most solutions were obtained in under a minute) and was more automatic being relatively insensitive to the initial guesses, which could therefore be programmed.

The results gave a clear definition of the boundary layer effect first discussed by Friedrichs and Stoker and the completeness of the data made possible an empirical determination of the width of the boundary layer as a function of the loading parameter.

3. 3. Plastic torsion of cylinders. The problem considered is that of torsion of a prismatic cylinder of a material with a well-defined yield such as mild steel. The work to be

summarized here, due to W. S. Dorn and the author, is still in progress and will be presented elsewhere with a complete discussion of method and results.

We give a brief review of the formulation of the problem[8]. Taking the x,y-plane in the cross-section and the z-axis parallel to the generators of the cylinder the St. Venant assumptions for the displacement components u, v, w are

$$u = -yz\Theta \ , \quad v = xz\Theta \ , \quad w = w(x,y,\Theta)$$

where Θ is the angle of twist per unit length of the cylinder. The strains are zero except for the shear strains given by

$$\gamma_x = -\Theta y + \frac{\partial w}{\partial x} \ , \qquad \gamma_y = \Theta x + \frac{\partial w}{\partial y} \ . \tag{7}$$

The components τ_x, τ_y of the shear stress can be obtained from a stress function $\psi(x,y,\Theta)$ by

$$\tau_x = \frac{\partial \psi}{\partial y} \ , \qquad \tau_y = -\frac{\partial \psi}{\partial x} \tag{8}$$

and equilibrium is thereby automatically satisfied for any function ψ vanishing on the boundary of the cross-section (assuming a simply connected region).

Making the assumption of an elastic-perfectly plastic

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[8] For a more detailed discussion of the plastic torsion problem see Prager [23].

material, Hooke's law provides the stress-strain relation as long as the condition $|\text{grad } \psi| = \sqrt{(\frac{\partial \psi}{\partial x})^2 + (\frac{\partial \psi}{\partial y})^2} < k$ is satisfied locally, where k is the yield stress in pure shear. Yielding takes place at points where the condition $|\text{grad } \psi| = k$ is satisfied. As θ increases from zero the solution is everywhere elastic until the yield condition is satisfied locally, following which plastic zones grow until ultimately (for an ∞ angle of twist) the entire cross-section would become plastic.

The mathematical problem is to determine for each θ a function $\psi(x,y,\theta)$ vanishing on the boundary, continuous with continuous first partial derivatives such that everywhere $|\text{grad } \psi| \leq k$ and such that wherever $|\text{grad } \psi| < k$, the elastic compatibility equation $\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 = -2G_o\theta$ is satisfied where G_o is the elastic modulus of rigidity.

The problem of solution is made difficult by the unknown boundary between the elastic and plastic regions and the distinct partial differential equations to be satisfied in these regions. No satisfactory systematic numerical procedure has been devised tc handle this problem. The difficulties arise because of the assumption of perfect plasticity. This assumption, which represents an approximation to the phenomenon of yield observed in metals like mild steel, is introduced to simplify tne mathematical theory in those problems where the entire material is yielding or the extent

of the plastic zone is known or satisfies certain à priori conditions such as those of limit analysis. However, in cases like the present one the assumption of perfect plasticity leads to unnecessary mathematical complications.

To eliminate these difficulties, one may use a plasticity theory for a material whose stress-strain curve is of the continuous transition variety. These curves have an initial elastic slope and bend continuously approaching either the yield stress k of perfect plasticity as an asymptote or increase beyond this to incorporate the effects of strain hardening.

Ramberg and Osgood [24] introduced a set of stress-strain curves depending on three parameters which fit a variety of metals and include work-hardening effects. For simple shear, the stress τ and strain γ (in the present notation) are related by an equation of the form

$$\gamma = \frac{1}{2G_o}\left[1 + \frac{\tau^{2n}}{k^{2n}}\right]\tau \qquad (9)$$

where n is a positive integer parameter, G_o and k are essentially the elastic rigidity and yield stress. For fixed n, k, G_o, the stress is a monotone increasing function of strain which flattens as it reaches k in value but increases beyond k in value as the strain increases. As $n \longrightarrow \infty$ the stress-strain curve approaches the broken line, flat yield curve of a material which behaves elastically

with shear modulus G_o up to the shear stress k and then yields under constant shear stress k. The larger n is, the more nearly is this relation approximated.

To appropriately generalize (9) to the case of several stress-components as in torsion, one may either proceed to formulate a deformation-type (finite) stress-strain law following Nadai [25] or a flow-type (differential) stress-strain law following Prager [23, 25, 26] and H. J. Laning (unpublished paper).[9]

Appropriate relations of the deformation type are given by

$$\begin{aligned} \gamma_x &= \frac{1}{2G_o}\left[1 + \frac{(\tau_x^2+\tau_y^2)^n}{k^{2n}}\right]\tau_x \\ \gamma_y &= \frac{1}{2G_o}\left[1 + \frac{(\tau_x^2+\tau_y^2)^n}{k^{2n}}\right]\tau_y \end{aligned} \qquad . \tag{10}$$

In terms of the reduced stress function $\bar{\psi} = \psi/k$, the use of (10) leads to the following partial differential equation for $\bar{\psi}$:

$$\begin{aligned} &(1+\bar{S}^{2n}+2n\bar{\psi}_{,x}^2\bar{S}^{2(n-1)})\bar{\psi}_{,xx}+4n\bar{\psi}_{,x}\bar{\psi}_{,y}\bar{S}^{2(n-1)}\bar{\psi}_{,xy} + \\ &+ (1+\bar{S}^{2n}+2n\bar{\psi}_{,y}^2\bar{S}^{2(n-1)})\bar{\psi}_{,yy} = -\frac{2G_o\theta}{k} \end{aligned} \tag{11}$$

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[9] The Ramberg-Osgood curves were also used by Huth [27] in a comparison of flow and deformation theories in plastic torsion. His calculations were carried out for a square cross-section using a low n value and a relatively coarse mesh.

where $\bar{S}^2 = \bar{\psi}_{,x}^2 + \bar{\psi}_{,y}^2$ and a comma followed by x or y indicates partial differentiation with respect to that variable. For a simply connected cross-section this equation is to be solved subject to the condition $\bar{\psi} = 0$ on the boundary. The numerical solution of this problem will be considered after the formulation which follows of the torsion problem in terms of a flow theory.

Appropriate stress-strain relations of the flow type are given by

$$2G_o\dot{\gamma}_x = \dot{\tau}_x + (2n+1)\frac{(\tau_x^2+\tau_y^2)^{n-1}}{k^{2n}}(\tau_x\dot{\tau}_x + \tau_y\dot{\tau}_y)\tau_x$$

$$2G_o\dot{\gamma}_y = \dot{\tau}_y + (2n+1)\frac{(\tau_x^2+\tau_y^2)^{n-1}}{k^{2n}}(\tau_x\dot{\tau}_x + \tau_y\dot{\tau}_y)\tau_y \qquad (12)$$

where the dot denotes differentiation with respect to θ which is assumed to increase monotonically and so plays the role of the time variable. Using these relations, the partial differential equation for $\bar{\psi}(x,y,\theta)$ becomes

$$[1+\bar{\psi}_{,x}^2\bar{S}^{2(n-1)}(2n+1)]\dot{\bar{\psi}}_{,xx} + [2(n+1)\bar{S}^{2(n-1)}\bar{\psi}_{,x}\bar{\psi}_{,y}]\dot{\bar{\psi}}_{,xy} +$$

$$+ [1+\bar{\psi}_{,y}^2\bar{S}^{2(n-1)}(2n+1)]\dot{\bar{\psi}}_{,yy} + [\bar{S}^{2(n-1)}(\bar{\psi}_{,x}\bar{\psi}_{,xx}+\bar{\psi}_{,y}\bar{\psi}_{,xy}) +$$

$$+ K\bar{\psi}_{,x}](2n+1)\dot{\bar{\psi}}_{,x} + [\bar{S}^{2(n-1)}(\bar{\psi}_{,x}\bar{\psi}_{,xy}+\bar{\psi}_{,y}\bar{\psi}_{,yy}) + K\bar{\psi}_{,y}](2n+1)\dot{\bar{\psi}}_{,y} =$$

$$= -\frac{G_o}{k}, \qquad (13)$$

where $K = 2(n-1)\bar{S}^{2(n-2)}[\bar{\psi}_{,x}^2\bar{\psi}_{,xx}+2\bar{\psi}_{,x}\bar{\psi}_{,y}\bar{\psi}_{,xy}+\bar{\psi}_{,y}^2\bar{\psi}_{,yy}]$.

This equation is subject to the boundary conditions $\bar{\psi}(x,y,\Theta)=0$ for a simply connected cross-section and for $\Theta \geq 0$.

The deformation theory equation for $\bar{\psi}$ is of the form

$$A\bar{\psi}_{,xx} + B\bar{\psi}_{,xy} + C\bar{\psi}_{,yy} = -\frac{2G_o\Theta}{k}$$

where A, B, C are functions of $\bar{\psi}_{,x}$ and $\bar{\psi}_{,y}$ and is a quasi-linear second order partial differential equation of elliptic type. A numerical procedure was devised at NYU by S. Schechter for handling general equations of this type and coded by R. Wernick for the IBM-704. This code was successfully used for the present problem. The following is a brief description of the procedure, the code and the results obtained.

Substituting differences for the derivatives in the equation one obtains a set of non-linear algebraic equations for the values of $\bar{\psi}$ defined on the points of a rectangular mesh. As in the previous example, an iterative procedure was used for the solution of these equations. Starting from an initial guess $\bar{\psi}^{(o)}$ at each point, the new value at a given mesh point was calculated as follows. The coefficients A, B, C of the difference equations are evaluated from the given $\bar{\psi}^{(o)}$ values. The differences for the second order terms are calculated from the $\bar{\psi}^{(o)}$ values as well except for the value at the given mesh point which is calculated so as to satisfy the equation. This value $\bar{\psi}^{*}$ constitutes

the uncorrected new iterate value at the point. To obtain the new iterate value $\bar{\psi}^{(1)}$ at the point, again a weighted average of the form

$$\bar{\psi}^{(1)} = \omega \bar{\psi}^{*} + (1-\omega)\, \bar{\psi}^{(o)}$$

is taken where the parameter ω now is found to give the best convergence for values greater than one, furnishing a so-called extrapolated value. Moving to the next mesh point proceeding in a prescribed order, say row by row, the coefficients A, B, C and second order differences are next computed from the old values $\bar{\psi}^{(o)}$ together with the new value of $\bar{\psi}^{(1)}$ calculated at the preceding point. In this way the latest iterate values are always used and each step furnishes an explicit value for the next iterate at the given mesh point.

For a convergence criterion, $\bar{\psi}^{(n+1)}$ and $\bar{\psi}^{(n)}$ are compared at each point of the mesh and the maximum of $|\bar{\psi}^{(n+1)} - \bar{\psi}^{(n)}|$ taken over all the points is taken as a measure of the error at the $(n+1)^{st}$ iteration or "sweep" of the mesh. In the present process the calculation was stopped when this quantity was less than 1×10^{-6}.

To gauge the size of the computational problem corresponding to a two dimensional problem of this complexity, the following data is informative. The Schechter-Wernick code was designed for more general equations in which the

coefficients A, B, C could be functions of $\bar{\psi}$, $\bar{\psi}_{,x}$, $\bar{\psi}_{,y}$, x and y and also for boundary conditions involving the normal derivative of $\bar{\psi}$ as well as $\bar{\psi}$. It also allowed for multiply-connected domains and permitted variable mesh spacing from point to point. The basic code for all this took up some 2800 words of memory. In addition, some 1500 words were required for general purpose subroutines such as input, output and edit. Over 400 words were used in our problem in routines for the calculation of the specific coefficient functions A, B, C, and for a 34 x 34 point mesh (the limit of the code) some 3400 words of storage were required. Thus, virtually the entire 8,196 word memory of a standard 704 was required.

The method was checked out against the case of a circular cross-section where the solution could be found analytically and was found to give excellent agreement. The calculations were then carried out for the case of a square cross-section where no analytic results are available but the character of the solution for the limiting case ($n \longrightarrow \infty$) of an elastic-perfectly plastic material is well known from the soap-film, sand-hill analogy of Nadai. According to this analogy, as θ, the angle of twist, is increased, the plastic zones will develop in from the centers of the sides of the squares and gradually fill out the four triangles formed by the diagonals of the square. The material in strips adjacent to the

diagonals constitutes a residual elastic region. In the plastic region, the stress intensity measured by S should remain constant equal to k.

The calculations bore this out and indeed for $n = 17$ with $2G_o\Theta/k = 16$, among the highest values of these quantities used, the plastic zones were sharply defined regions of essentially constant S.

With a 20×20 mesh and $n = 13$, starting from an arbitrary initial guess of $\bar{\psi}^{(o)} \equiv 1/8$ at all points, convergence to 1×10^{-6} required about 50 or 60 sweeps at 6 seconds per sweep and so took about 6 minutes for a given Θ value. With $n = 17$ (so that the coefficients A, B, C involved 34^{th} powers of the differences for $\bar{\psi}_{,x}$ and $\bar{\psi}_{,y}$, which were calculated by multiplications rather than logarithms) the time went up to about 10 seconds per sweep for a total time of about 10 minutes for a whole calculation. Starting from a previous solution for a smaller Θ value for the initial guess $\bar{\psi}^{(o)}$, the time was cut by about one-half in each case.

For large values of $2G_o\Theta/k$, above 16, the elastic zone along the diagonals became too small for adequate definition using a 20×20 mesh in the square and a 34×34 mesh was used, which was at the limit of the given code. With this mesh, values of $2G_o\Theta/k$ as high as 25 were reached. However, with this number of points the

time per sweep went up, for $n = 17$, to about 25 seconds and solution time for a given angle of twist to 20 to 30 minutes, revealing the enormous amount of calculation involved.

It is intended to compare the results of this solution of the deformation theory equation with a solution of the flow equation (13) and to further compare predictions of the two types of theories for other cross-sections. Such comparisons, followed by further comparisons in two-dimensional problems, it is hoped will lead to the selection of a valid stress-strain relation covering through parameters a number of metals, which can be used as a basis for complete elastic-plastic analyses of continuous structures using computers.

The flow equation (13) considered as one in the three independent variables x, y and θ is of third order and is essentially parabolic. Equations of this type in which the "time" derivative is not given explicitly but appears implicitly as the solution of a space variable elliptic type problem is encountered also in meteorology problems. The numerical problem of solution is a formidable one and there is as yet comparatively little known theoretically or empirically about methods of solution. A method which suggests itself is to begin with a small θ value, use the elastic solution to calculate the coefficients of the

highest order derivatives and then solve the linear 2nd order problem for $\dot{\bar{\psi}}$. These values can be used in a time step to advance the value of $\bar{\psi}$ to the next θ value and then repeat the process. Probably accuracy will be improved by some form of averaging of time step values. It does not appear that there is any numerical stability problem here as there is in time dependent problems of the heat-flow type.

C. Conclusions and Beginnings.

Any number of self-evident things can be said about the future importance of computers in mechanics or any other science. To avoid saying these again is difficult. However, the facts speak for themselves. The number of computers available is already large and growing constantly. The number of people to whom programming and coding and operating a computer is routine is growing at a pace where in a short time it will be as familiar, if not quite as portable, an aid to the student as the slide rule. The speed and capacity of computers is being enlarged (on the order of a factor of 10) with each generation of computers (on the order of 5 years) to the point where it is safe to assume that with a practical amount of analysis, time and money we can soon solve any of the mechanics problems we currently can formulate. Automatic programming routines will allow the user to solve his problems even on the largest computers with relative ease and a minimum of effort.

What is needed is intelligent use of these machines. As has been indicated there is a place for problem solving by simple application of existing mathematical subroutines appropriately combined. There is also a place, although a disappearing one, for simple adaptation of methods and formulas of an older era in structural mechanics when approximation was dictated by the computational limitations of human mind

or hand. The challenge to be met by today's expert is to formulate his problem and his method of solution to the limits of accuracy of existing computers and ultimately to formulate as complete and accurate a description of the problem as is possible based on existing physical knowledge.

By virtue of the accuracy and completeness of the results obtainable to problems properly formulated and solved on computers revolutionary new procedures become feasible. For example, the computer may be used to experiment on a structure in varieties of ways and with a control of variables never possible in a physical laboratory. If the model programmed into the computer has been conceived in broad enough terms to include the essential parameters so as to fully describe the processes involved, every conceivable combination of loads, material properties, etc., can be investigated and with a speed and simplicity otherwise impossible. The example of the spherical shell caps cited previously provided just such an opportunity. Shells were picked and loaded to buckling and beyond as they suggested themselves to the experimenter almost as would have been done in an actual laboratory, giving measurements of stresses and deflection impossible to obtain on actual specimens.

Again, the computer may be used to select a design by comparing the behavior of a variety of structures of a many

parameter family. An outstanding application of this approach today lies in the field of design of turbine blades.

A basic application of the computer will be, as in the plastic torsion problem, to compare the predictions of competing theories in critical experiments.

In all of this it is clear that new methods of analysis must be developed; mathematical procedures must be suited to the mechanical problems concerned and designed for efficient machine use.

At the present time, the most standard tool of numerical analysis in problems in continuum mechanics is the difference equation approximation to the differential equation. This leads to a system of simultaneous algebraic equations, which, even in two dimensions for a single function for say a 10×10 mesh is of order 100 and in three dimensions for a $10 \times 10 \times 10$ mesh is of order 1000. It is true that even in the non-linear case, iterative procedures of solution (such as described for the plastic torsion problem) exist wherein the computational complexity at each step (say in going from point to point) is independent of the total number of unknowns. However, the stability of the procedure and the accuracy of the results becomes increasingly doubtful as the order grows. One is, therefore, faced with the dilemma that by putting more points into the mesh to

improve in principle the accuracy of one's solution, one loses accuracy for reasons of the accumulation of errors in the number of numerical operations required for solution using the given method of solution.

Ideally what is needed are methods of solution in which greater accuracy is obtained as one increases the amount of arithmetic in calculating the solution. This is true of the Monte Carlo methods [28] which simulate the physical processes and wherein the accuracy of the results increases as more trials are run or more games are played. For processes like diffusion where a physical model can be set up describing the probabilistic behavior of individual particles the Monte Carlo method is most suitable and is being used successfully. Unfortunately, the deterministic (or what we today consider deterministic) problems of solid mechanics do not in their physical models suggest natural Monte Carlo procedures. One might indeed devise Monte Carlo procedures for the solution of the specific differential equations of a given mechanics problem, but this is not appealing as it runs counter to the spirit of the method which is used partly to avoid the differential equation description. Still it may be that a more basic microscopic formulation of mechanics problems in terms of fundamental physical laws will one day be possible and that at that time the Monte Carlo methods will perhaps be the natural ones.

Lacking Monte Carlo procedures and limited in scope by difference equations there are few guideposts to methods for solving complex structures in three dimensions. One suggestion which the author should like to make, and whose practicality has yet to be tested, is a return to the direct methods of the calculus of variations -- in modern dress. Traditionally, for linear problems, this has meant approximating the solution by a linear combination of m selected functions and then minimizing the appropriate energy expression with respect to the coefficients of these functions leading to a system of linear equations of order m. The result is the best combination of the selected functions.

To describe an alternative procedure let us consider the specific example of finding the displacements u, v, w throughout an elastic solid under prescribed surface tractions and displacements. We replace the continuous body by a rectangular lattice of n points and seek the displacements at these points, a total of $3n$ unknowns (this would be an enormous number, 3000 in the case of even a modest $10 \times 10 \times 10$ mesh of points). Next, the potential energy is expressed in terms of these unknowns. Using first differences for strains and combinations of these for stresses this yields a quadratic expression in $3n$ unknowns to be minimized. The boundary conditions

can be used to reduce the number of unknowns in this expression or can be left (particularly when the conditions are on stresses) as subsidiary linear conditions on some of the unknowns.

If one next proceeded to use the calculus to find the conditions for a minimum, one would be led to a system of 3n linear equations, more or less, for the unknowns. These would, of course, comprise a system of approximating difference equations for the partial differential equations governing the displacement functions u, v, w. In fact, this is recognized as a useful and consistent way of deriving such equations. However, this procedure leads to the problem of solving a linear system of large order which is what we wish to avoid.

It is suggested that instead of the above procedure, attention be concentrated directly on the problem of minimizing the quadratic function in the unknowns. Any linear elasticity problem can of course always be reduced to this form. The problem of finding direct methods of minimizing a quadratic function subject to linear constraints is one that is currently receiving considerable attention under the name of quadratic programming [29-35]. Various algorithms have been devised for proceeding step-wise to a solution of the problem. The most recent and promising of these due to Wolfe [35] is based on a simple modification of

the simplex method of linear programming. These methods, however, are intended for problems wherein the variables are subject to linear inequalities and hence, which cannot be reduced via the calculus to the solution of linear equations. The programming techniques are usually deemed unnecessary where no inequalities are present as is the case in the elasticity problem. However, the algorithm devised by Wolfe can be applied to this case in principle to find after a finite number of iterations the set of values for u, v, w which minimizes the potential energy. How this procedure compares in time and accuracy with the solution of linear equations procedure remains to be studied. At the present time the IBM-704 code prepared by Wolfe for the quadratic programming problem and based on the previously cited code of Orchard-Hayes, Judd and Cutler, limits the sum of unknowns and subsidiary conditions to 255. Thus, without modification it would only be feasible to study at most two dimensional elasticity problems and even here the limitations are stringent since the procedure would more than double the number of physical unknowns to be determined. (However, for structures like frames the number of unknowns allowed would seem ample.)

Clearly this is a case where the existing mathematical procedure should be studied in the light of the given mechanics problem. There is no question that larger machines

will raise the bounds on the number of unknowns in the method as it stands to practical levels. On the other hand, by tailoring the method to the problem there is a good chance that the matrices can be reduced in size to the point where good use of present machines can be made with essentially this type of solution of elasticity problems.

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